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by
Robert M. Saltzman
and Frederick S. Hillier

TECHNICAL REPORT SOL 88-11

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Abstract

The Role of Ceiling Points in General Integer Linear Programming

Robert M. Saltzman and Frederick S. Hillier

Stanford University, 1988

This report examines the role played by several kinds of ceiling points in solving the pure, general integer linear programming problem *(ILP)*. While no assumptions are made concerning the structure or signs of the data of the problem, it is assumed that the feasible region for *(ILP)* is non-empty and bounded. A ceiling point with respect to a single constraint may be thought of as an integer solution on or close to the boundary of the feasible region defined by the constraint. The definition of a ceiling point with respect to a single constraint is extended to take multiple constraints into consideration simultaneously, defining what is called a feasible ceiling point. It is shown that the set of all feasible ceiling points contains at least one optimal solution for *(ILP)*. A related class of solutions called feasible 1-ceiling points is also characterized and shown to contain *all* optimal solutions for *(ILP)*. Moreover, 1-ceiling points are computationally easier to identify than ordinary ceiling points and may be sought with respect to one constaint at a time. It is also demonstrated that solving *(ILP)* requires only enumerating feasible 1-ceiling points with respect to a subset of all functional constraints.

1. Introduction and Overview

In this report we consider the role played by ceiling points in solving the pure, general integer linear programming problem (ILP) in m constraints and n variables x_j, $j = 1, ..., n$, whose form is

$$\begin{aligned} \text{maximize } & c^T x = z \\ \text{subject to } & Ax \leq b \\ & x \geq 0, \quad x \text{ integer}, \end{aligned} \qquad (ILP)$$

where $A \in \Re^{m \times n}$, $b \in \Re^m$ and $c \in \Re^n$. All the data $\{A, b, c\}$ are assumed to be rational numbers, but they are unrestricted in sign. The problem is "pure" in that all of the variables are required to take on nonnegative integer values. It is "general" in the sense that the variables may take on any nonnegative integer values permitted by $Ax \leq b$, as opposed to being restricted to 0 or 1 (the binary case). An important additional assumption is that no implicit or explicit equality constraints are used to define the feasible region $FR \equiv \{x \geq 0 | \ Ax \leq b\}$. This implies that any $m' \times n'$ subsystem of equality constraints, where $m' < n'$, has already been reexpressed by solving for m' of the variables in terms of the other $n' - m'$ variables and substituting for these m' variables in all of the remaining inequality constraints, yielding a problem consisting only of inequality constraints.

Our goal is to develop a solid understanding of the nature and utility of ceiling points. In the next section, ceiling points are defined and some of their properties explored. Section 3 extends these basic definitions to those of d-ceiling points, which are computationally easier to identify than ordinary ceiling points. In Section 4, we consider what constraints are most critical in the search for an optimal solution, while the restrictiveness of assumptions made in various theorems is examined in Section 5. Section 6 summarizes the key concepts discussed in this report using an example in the plane.

2. Characterizing Ceiling Points

Before formalizing the intuitive concept of a ceiling point as an integer solution on or close to the surface of the feasible region for the linear programming relaxation (LP_R) of (ILP), a few conventions are noted. Throughout the report, constraint (i) may refer to a functional or nonnegativity constraint of the form $a_i^T y \leq b_i$. If it refers to the j^{th} nonnegativity constraint, all coefficients a_{ij} are 0, except for the j^{th} which is -1, and the corresponding right hand side b_i is 0. Also, the term "direction" will be used to mean a direction in $\Re^n$ which is parallel to one of the n coordinate axes. The first ceiling point definition considers an integer point satisfying a single constraint i with little or no slack.

Definition 1 An integer vector x is a *ceiling point with respect to the i^{th} constraint*, denoted $x = CP(i)$, if

(1) $a_i^T x \leq b_i$, and

(2) $a_i^T x + |a_{ij}| > b_i$, for each $j = 1, ..., n$.

The first part of the definition simply means that x satisfies the i^{th} constraint; the second part of the definition, called the ceiling point condition, implies that taking a unit step from x toward the i^{th} constraining hyperplane in every direction results in an infeasible point. Letting $x_j^+ \equiv x + e_j$ and $x_j^- \equiv x - e_j$ represent the points which are a unit step away from x in the plus j and minus j directions, respectively, we can reexpress (2) from our definition as follows. If $a_{ij} > 0$ then x_j^+ violates constraint (i), whereas if $a_{ij} < 0$ then x_j^- violates constraint (i), for all j. If $a_{ij} = 0$ then changing the j^{th} component of x does not affect the feasiblility of x with respect to constraint (i). †

† It may be interesting to compare Definition 1 with that first given by J. Wilson [Wi, Definition 1]. Assuming that $a_{i1} \geq a_{i2} \geq \ldots \geq a_{in} > 0$, $b_i > 0$, and all data are integer, he defines a nonnegative integer vector x to be a ceiling point with respect to a single constraint $a_i^T y \leq b_i$ if

(1) $a_i^T x \leq b_i$,

Thus, according to the ceiling point condition (2) of the definition, no ceiling points with respect to (i) alone exist when any $a_{ij} = 0$. This suggests the need for a second definition which considers not just one constraint, but all of the nonnegativity and functional constraints of (ILP) simultaneously.

Definition 2 An integer vector x is a *ceiling point with respect to the feasible region* $FR \equiv \{x|\ Ax \leq b, x \geq 0\}$, denoted $x = CP(FR)$, if

(1) $x \in FR$, and

(2) $\exists i : a_i^T x + |a_{ij}| > b_i$, for each $j = 1, ..., n$.

In this case, $x = CP(FR)$ indicates that x is feasible and for every direction j either x_j^+ or x_j^- (or both) violate at least one functional or nonnegativity constraint. Notice that $x = CP(i)$ and $x \in FR$ imply that $x = CP(FR)$. However, the converse is not true, *i.e.*, $x = CP(FR)$ does not imply that $x = CP(i)$, for some (i), as illustrated by point **x** in Figure 1.

Ceiling points in an integer linear program play a role analogous to that of corner-point feasible solutions in a linear program.

Lemma 1a Every extreme point of the convex hull of feasible integer solutions for (ILP) is a $CP(FR)$.

Proof: First note that all extreme points of the convex hull of feasible integer solutions are themselves feasible integer solutions (since the vertices of the convex hull of a finite set

(2) $a_i^T x + a_j \geq b_i + 1$, for $j = 1, ..., n$, and
(3) $x_{j+1} > 0 \Rightarrow a_i^T x + a_j - a_{j+1} \geq b_i + 1,\ \ 1 \leq j < n$.
In contrast, Definition 1 makes no assumptions on the relative magnitudes, signs or integrality of the coefficients, nor does it require the vector x to be nonnegative. Wilson's goal is to first locate n ceiling points satisfying the i^{th} constraint with equality, and then use these points to construct the minimal inequality corresponding to this constraint, *i.e.*, to strengthen this linear integer inequality by reducing its coefficients as much as possible.

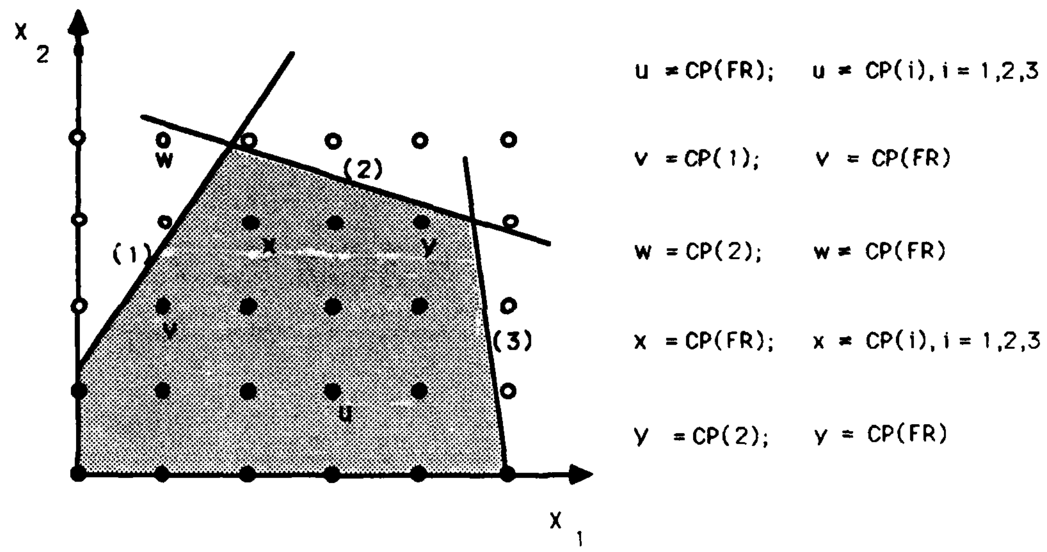

Figure 1. Examples of Ceiling Points.

of points are a subset of the points from which the convex hull is formed [PS, p. 37]). Now suppose that x is such an extreme point which is not a $CP(FR)$. Then there is a direction j such that both x_j^+ and x_j^- are feasible. Hence, $x = (x_j^+ + x_j^-)/2$, contradicting the extreme nature of x. ∎

Because they tend to be closer to the boundary of FR than non-ceiling points, ceiling points in the "interesting" portion of FR are likely candidates for having the highest objective function values. This idea has motivated our interest in ceiling points: they seem to be the points on which we should focus our attention. Indeed, we can show that if an optimal solution x^* for (ILP) exists, either x^* itself is a $CP(FR)$ or there exists another optimal solution which is a $CP(FR)$.

Theorem 1 Suppose the set of feasible solutions for (ILP) is non-empty and bounded. Then there exists an optimal solution for (ILP) which is a $CP(FR)$.

Proof: Since (ILP) possesses at least one feasible integer solution and is bounded, it possesses an optimal solution. Furthermore, the convex hull of feasible integer solutions is non-empty. By a well-known theorem (see [HL, p. 54] or [Si, p. 91]) an optimal solution for a linear programming problem occurs at one of the extreme points of its feasible region. Similarly, in an integer linear program, an optimal solution occurs at an extreme point of the convex hull of feasible integer solutions. Since all such extreme points are $CP(FR)$'s, by Lemma 1a, it follows that at least one optimal solution for (ILP) is a feasible ceiling point, a $CP(FR)$. ∎

The analogy between ceiling points and extreme points, however, should not be carried too far. First notice that not every ceiling point is an extreme point of the convex hull of feasible integer solutions for (ILP) as evidenced by the ceiling point $\mathbf{v}$ in Figure 1. Moreover, while only a finite number of extreme points in the feasible region of (LP_R) may exist, the feasible region of the associated (ILP) may possess an infinite number of ceiling points if it is unbounded. Even if the feasible region is bounded, the number of $CP(FR)$'s is likely to be much larger than the number of extreme points in the associated (LP_R). Nonetheless, the main implication of the theorem is that an (ILP) may be solved by enumerating and comparing the values of all $CP(FR)$'s: the ceiling point(s) with the largest objective value optimally solve(s) the problem. Also, if a complete enumeration is not possible, it might be useful to concentrate one's effort on locating ceiling points relatively near $\bar{x}$, where any feasible solution found is apt to be a high-quality and sometimes even optimal solution. Locating such ceiling points is the goal of the Heuristic Ceiling Point Algorithm described in a subsequent report.

Corollary 1b Suppose the set of feasible solutions to (ILP) is non-empty and bounded. Further suppose all components of c differ from zero. Then every optimal solution for

(ILP) is a $CP(FR)$.

Proof: Since (ILP) possesses at least one feasible integer solution, it also possesses an optimal solution. Let x^* be an optimal solution for (ILP) which is not a $CP(FR)$. Then, since x^* is feasible, there must exist at least one direction j such that both x_j^{*+} and x_j^{*-} are feasible. Consider any such direction j. Since $c_j \neq 0, \forall j$, either x_j^{*+} or x_j^{*-} has a strictly superior objective value to that of x^*, (because either $c^T x_j^{*+} = c^T x^* + c^T e_j = c^T x^* + c_j > c^T x^*$, or $c^T x_j^{*-} = c^T x^* - c^T e_j = c^T x^* - c_j > c^T x^*$), contradicting the optimality of x^*. ∎

A few more observations can be made about the utility of ceiling points, or lack thereof. When some $c_j = 0$, the proof of Corollary 1b implies that there may exist non-ceiling points which are optimal. On the other hand, if the optimal solution for (ILP) is unique, it must be a $CP(FR)$ by Theorem 1. However, we cannot assume a priori that the optimal solution is unique nor is it realistic to assume that all c_j differ from 0. For instance, in fixed-charge problems, the fixed-charge (0-1) variables sometimes have a zero cost component. Consequently, we have found it beneficial to relax the ceiling point definitions given above.

3. d-Ceiling Points and Their Properties

From a computational standpoint, identifying a $CP(FR)$ is difficult: in order to satisfy the ceiling point condition, (part (2) of the definition), a constraint must be violated as a result of either increasing or decreasing *every* component by one unit. We now describe two sets of points which are formed by relaxing the ceiling point conditions of Definitions 1 and 2, respectively, so that a constraint violation does not have to occur in every direction from the ceiling point. Consequently, these sets are easier to enumerate than their respective ceiling point counterparts.

Definition 3 For an integer $d \in [1, n]$, an integer vector x is a *d-ceiling point with respect to the i^{th} constraint*, denoted $x = d\text{-}CP(i)$, if

(1) $a_i^T x \leq b_i$, and

(2) $a_i^T x + |a_{ij}| > b_i$, for each of d distinct indices $j = j_1, ..., j_d$.

In other words, $x = d\text{-}CP(i)$ means x satisfies the i^{th} constraint, and taking a unit step from x toward the i^{th} constraining hyperplane in directions $j = j_1, ..., j_d$ (and perhaps in some additional directions) results in an infeasible point. For example, if $a_{ij_1} > 0$ then $x_{j_1}^+$ violates constraint (i), whereas if $a_{ij_1} < 0$ then $x_{j_1}^-$ violates constraint (i). As with ordinary ceiling points, Definition 3 may be extended to cover a set of constraints which comprise the feasible region FR of an (ILP).

Definition 4 For an integer $d \in [1, n]$, an integer vector x is a *d-ceiling point with respect to the feasible region FR*, denoted $x = d\text{-}CP(FR)$, if

(1) $x \in FR$, and

(2) $\exists i : a_i^T x + |a_{ij}| > b_i$, for each of d distinct indices $j = j_1, ..., j_d$.

In this case, $x = d\text{-}CP(FR)$ indicates that x is feasible and in directions $j = j_1, ..., j_d$ either x_j^+ or x_j^- (or both) violate at least one functional or nonnegativity constraint. It should be clear that a point which is a $\bar{d}\text{-}CP(i)$ for a particular $\bar{d}$ is also a $d\text{-}CP(i)$ for every integer $d \in [1, \bar{d} - 1]$. The same is true for $d\text{-}CP(FR)$'s. The points in Figure 1 may be recast as $d\text{-}CP(i)$'s as follows: $\mathbf{v} = 2\text{-}CP(1)$, $\mathbf{w} = 2\text{-}CP(2)$, $\mathbf{x} = 1\text{-}CP(1)$ and $\mathbf{x} = 1\text{-}CP(2)$, while $\mathbf{y} = 2\text{-}CP(2)$ and $\mathbf{y} = 1\text{-}CP(3)$. It is worth emphasizing that a single point may be a $d\text{-}CP(i)$ with respect to more than one constraint, especially if d is allowed to vary. As with ceiling points, $x = d\text{-}CP(i)$ and $x \in FR$ imply that $x = d\text{-}CP(FR)$. Again referring to Figure 1, the points $\mathbf{v}, \mathbf{x}$, and $\mathbf{y}$ all are $d\text{-}CP(FR)$'s, with $d = 2$.

We now wish to clarify the relationships between the various sets of ceiling points introduced. The definitions indicate that an $n\text{-}CP(i)$ is equivalent to a $CP(i)$, as is an

n-$CP(FR)$ to a $CP(FR)$. However, when $d < n$, the ceiling point condition for a d-$CP(i)$ is less restrictive than that for a $CP(i)$, as it is for a d-$CP(FR)$ in comparison to that for a $CP(FR)$. This implies that the cardinality of the set of ordinary ceiling points with respect to a constraint (i) is no larger than the cardinality of the set of d-ceiling points with respect to that same constraint, for any specific integer $d \in [1, n]$. Just as the set of $CP(i)$'s with respect to a particular (i) is a subset of the set of that constraint's d-$CP(i)$'s, the set of $CP(FR)$'s is a subset of the set of all d-$CP(FR)$'s. The following lemma summarizes some of the key relationships between the four types of ceiling points, and is illustrated for $d = 1$ by Figure 2.

Lemma 1c For any integer $d \in [1, n]$ the following relationships exist among the 4 sets of ceiling points identified by Definitions 1 - 4:

(a) $\{x|\ x = CP(FR)\} \subseteq \{x|\ x = d\text{-}CP(FR)\}$

(b) $\bigcup_i \{x|\ x = CP(i)\} \subseteq \bigcup_i \{x|\ x = d\text{-}CP(i)\}$

(c) $\{x|\ x = d\text{-}CP(FR)\} \subseteq \bigcup_i \{x|\ x = 1\text{-}CP(i)\}$

(d) $\bigcup_i \{x|\ x = \text{feasible } d\text{-}CP(i)\} \subseteq \{x|\ x = d\text{-}CP(FR)\}$

Proof: (a) From the ceiling point conditions of Definitions 2 and 4, we have $\{x|\ x = 1\text{-}CP(FR)\} \supseteq \{x|\ x = 2\text{-}CP(FR)\} \supseteq \ldots \supseteq \{x|\ x = n\text{-}CP(FR)\} = \{x|\ x = CP(FR)\}$ Thus, every $CP(FR)$ is a d-$CP(FR)$. Figure 2 depicts this for the case where $d = 1$: the set of $CP(FR)$'s, labelled as box 3, is competely contained within the set of 1-$CP(FR)$'s, labelled as box 4.

(b) From the ceiling point conditions of Definitions 1 and 3, $\{x|\ x = 1\text{-}CP(i)\} \supseteq \{x|\ x = 2\text{-}CP(i)\} \supseteq \ldots \supseteq \{x|\ x = n\text{-}CP(i)\} = \{x|\ x = CP(i)\}$. Since this holds for all constraints individually, it also holds for the union over all constraints. This is shown for $d = 1$ in Figure 2 by the box labelled 2 being completely contained within box 1, the union of the set of 1-$CP(i)$'s.

(c) Let x be any *d-CP(FR)*. Then there is at least one direction such that either x_j^+ or x_j^- violates some constraint. Thus, x is a 1-$CP(i)$ for at least one (i). However, equality in (c) does not hold since a 1-$CP(i)$ may violate a constraint other than (i). With $d = 1$, the box labelled 4 in Figure 2 is completely contained within box 1.

(d) Let x be any feasible *d-CP(i)*. Then x satisfies the definition of *d-CP(FR)*. However, equality in (d) does not necessarily hold for $d > 1$ because a $CP(FR)$ need not be a *d-CP(i)* with respect to any particular constraint (as illustrated for $d = 2$ by the point $\mathbf{x}$ in Figure 1). With $d = n$, the union over (i) of the set of all *feasible n-CP(i)*'s is shown in Figure 2 as the intersection of boxes 2 and 3, labelled box 5. ∎

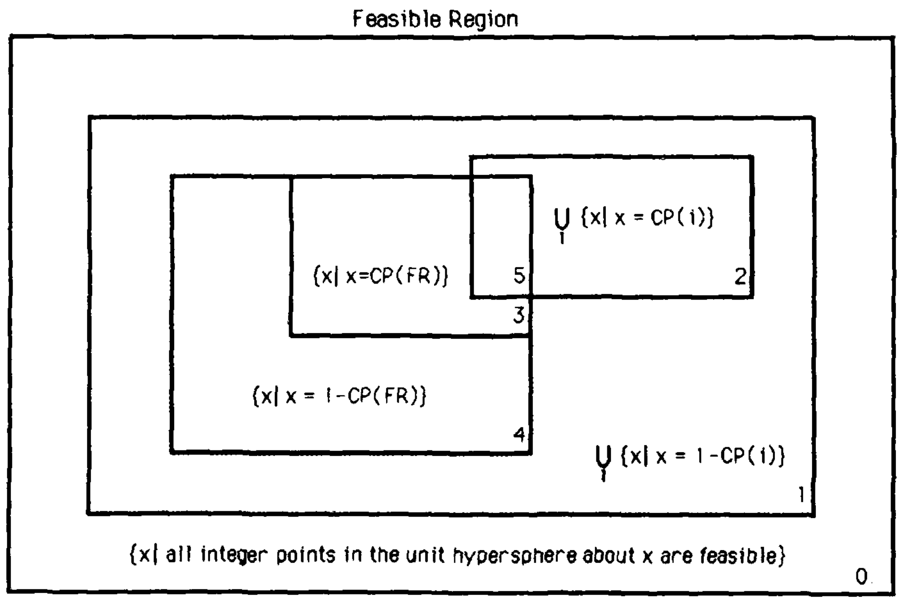

Figure 2. Relationships Between the 4 Types of Ceiling Points (Lemma 1c).

In general, the larger the magnitude of d in *d-CP(i)*, the closer a d-ceiling point is to the surface of the feasible region. However, an *n-CP(i)* is not necessarily a more

desirable solution than a 1-$CP(i)$. First, the n-$CP(i)$ may be infeasible while the 1-$CP(i)$ is feasible. Second, assuming feasibility, the objective function value of these two integer solutions depends more upon their location in the feasible region relative to the objective function hyperplane than it does upon d. What is important to note is that even a feasible 1-$CP(i)$, *i.e.*, a 1-$CP(FR)$, may be an optimal solution, as demonstrated in the next theorem.

Theorem 2 Suppose the set of feasible solutions for (ILP) is non-empty and bounded. Let $S_c \equiv \{j|\ c_j \neq 0\}$ be the support of c. Then every optimal solution for (ILP) is a d-$CP(FR)$, where $d \geq |S_c|$.

Proof: Since (ILP) possesses at least one feasible integer solution and is bounded, it also possesses an optimal solution. Let x^* be an optimal solution which is not a $|S_c|$-$CP(FR)$. Then, for at least one $j \in S_c$, *both* x_j^{*+} and x_j^{*-} are feasible. However, since $c_j \neq 0, \forall j \in S_c$, either x_j^{*+} or x_j^{*-} has a strictly superior objective value to that of x^*, contradicting the optimality of x^*. ∎

In particular, if $|S_c| = 1$, then all optimal solutions for (ILP) are 1-$CP(FR)$'s. This fact should not be too surprising since $\{x|\ x = 1\text{-}CP(FR)\}$ excludes only those points which are "interior" to the feasible region in the sense that all *integer* points within the unit hypersphere about them are feasible. (These are the points in the box labelled 0 in Figure 2 which are not in any other box.) While the set of $CP(FR)$'s contains the set of *vertices* of the convex hull of all feasible integer solutions to (ILP), the set of 1-$CP(FR)$'s contains all integer points on the *surface* of this convex hull, including its vertices. Consequently, another way of solving (ILP) is to enumerate the feasible elements of $\bigcup_i \{x|\ x = 1\text{-}CP(i)\}$ because among them are all the 1-$CP(FR)$'s for (ILP). The advantage of taking this route is that we can focus on finding 1-$CP(i)$'s with respect to one constraint at a time, while checking simultaneously for feasibility. This is the central approach of the algorithms presented in subsequent reports for solving an integer linear program.

When $|S_c| > 1$, can we solve (ILP) by enumerating all feasible $|S_c|$-$CP(i)$'s with respect to one constraint at a time? Unfortunately, if $|S_c| > 1$ and we search only for feasible $|S_c|$-$CP(i)$'s, we may fail to find some $|S_c|$-$CP(FR)$'s, by Lemma 1c(d). Moreover, it is possible that the *only* optimal solution for an (ILP) is a feasible 1-$CP(i)$ which is not also a feasible d-ceiling point $(d > 1)$ with respect to any single constraint. In this case, enumerating only feasible $|S_c|$-$CP(i)$'s will fail to yield the optimal solution. For example, consider the following simple (ILP):

$$\max\{x_1 + x_2 + x_3 | x_j \leq 4.5, j = 1,2,3; x \geq 0\}.$$

The integer solution $y = (4,4,4)$ is a 3-$CP(FR)$, yet y is a 1-$CP(i)$ with respect to each upper bound constraint individually. In fact, no 2-$CP(i)$'s or 3-$CP(i)$'s exist. Here, y is the unique optimal solution, and it would not be found by searching only for $|S_c|$-$CP(i)$'s. Thus, if we take the approach of seeking d-$CP(i)$'s with respect to one constraint at a time, we must take $d = 1$.

4. Search Constraints

Suppose we have an algorithm G for finding 1-$CP(i)$'s with respect to a given constraint (i). Then Theorem 2 indicates that to solve (ILP) we need to apply our algorithm G to each functional and nonnegativity constraint. The purpose of this section is to demonstrate that we only need to search for 1-$CP(i)$'s with respect to a subset of all the constraints. Also, this set of "search constraints" is not difficult to identify. The next theorem indicates that we may exclude from the set of search constraints all of the nonnegativity constraints because if there exists an optimal solution which is a 1-ceiling point with respect to a *nonnegativity* constraint, there also exists an optimal solution which is a 1-ceiling point with respect to a *functional* constraint.

Theorem 3 Suppose the set of feasible solutions for (ILP) is non-empty and bounded. Further suppose that (ILP) is not trivial to solve (by trivial we mean that if $x = 0$ is feasible then it is optimal). Then (ILP) possesses an optimal solution which is a feasible 1-$CP(i)$, where (i) is a *functional* constraint.

Proof: Let x^* be any optimal solution for (ILP). Assume that neither x^* nor any other optimal solution is a 1-$CP(i)$, where (i) is some functional constraint.

For all $j = 1, ..., n$ there are three possibilities:

Case I. $c_j > 0$: Neighboring solution x_j^{*+} violates no functional constraints since $x^* \neq$ 1-$CP(FR)$, nor does it violate any nonnegativity constraints because $x^* \geq 0$. Furthermore, $c^T x_j^{*+} > c^T x^*$, contradicting the optimality of x^*.

Case II. $c_j = 0$: Solutions along a ray emanating from x^* all have the same objective value, *i.e.*, $c^T x^* = c^T(x^* + e_j) = ... = c^T(x^* + Ke_j)$. Eventually, for some K large enough, $x^* + Ke_j$ violates a functional constraint (i_j) by the boundedness assumption. Then $x^* + (K-1)e_j$ is a feasible, optimal 1-$CP(i_j)$, contradicting the hypothesis.

Case III. $c_j < 0$: If neighboring solution x_j^{*-} is feasible, then $c^T x_j^{*-} > c^T x^*$, contradicting the optimality of x^*. Thus, the only remaining possibility is that x_j^{*-} is infeasible, implying that $x_j^* = 0$, which is considered next.

Since these cases hold for every direction j, we are left with $x_j^* = 0$ for all j, *i.e.*, $x^* = 0$ is the only optimal solution for an (ILP) with $c < 0$. But this is precisely the situation ruled out by the theorem's suppositions: If $x = 0$ is a feasible solution for the (ILP) $\max\{c^T x \mid Ax \leq b,\ x \geq 0,\ c \leq 0\}$, then $x = 0$ must be optimal. Therefore, either x^* itself or another optimal solution is a 1-$CP(i)$ with respect to some functional constraint (i). ∎

We now demonstrate that, at least in $\Re^2$, we need not search for 1-$CP(i)$'s with respect to any constraint that does not intersect the unit cube with all-integer vertices containing $\bar{x}$, an optimal solution for (LP_R). To help show this, let $\bar{A} \equiv \{i \mid a_i^T \bar{x} = b_i\}$ be

the set of constraints binding at $\bar{x}$, and $C_1 \equiv \{x|\ a_i^T x \leq b_i, \forall i \in \bar{A}\}$ be the cone formed by the extreme rays of FR emanating from $\bar{x}$. Also let $K \equiv \{j|\ \bar{x}_j = \text{integer}\}$ be the set of indices of integer-valued components of $\bar{x}$, $\kappa \equiv |K|$ and $UHC[\bar{x}]$ be any of the 2^κ unit hypercubes in $\Re^n$ with all-integer vertices which contain $\bar{x}$. If $\kappa = 0$, *i.e.*, $\bar{x}_j \neq$ integer, $\forall j$, then there is a unique $UHC[\bar{x}] \equiv \{x \in \Re^n|\ \lfloor \bar{x}_j \rfloor \leq x_j \leq \lceil \bar{x}_j \rceil, \forall j\}$. If $\kappa = 1$, *i.e.*, $\bar{x}_l =$ integer, for some l, then there are two corresponding $UHC[\bar{x}]$'s: $\{x \in \Re^n|\ \bar{x}_l - 1 \leq x_l \leq \bar{x}_l;\ \lfloor \bar{x}_j \rfloor \leq x_j \leq \lceil \bar{x}_j \rceil, \forall j \neq l\}$ and $\{x \in \Re^n|\ \bar{x}_l \leq x_l \leq \bar{x}_l + 1;\ \lfloor \bar{x}_j \rfloor \leq x_j \leq \lceil \bar{x}_j \rceil, \forall j \neq l\}$. If $\kappa > 1$, then there exist 2^κ $UHC[\bar{x}]$'s defined in the obvious manner. Since it is assumed that $\bar{x}$ is not all-integer, the scalar $\kappa \in \{0, 1, ..., n-1\}$.

Theorem 4 Suppose an (ILP) with $n = 2$ is non-empty and bounded. Further suppose that $c \neq 0$ and $\bar{x}$ is an optimal solution for (LP_R). Then every optimal solution for (ILP) is a feasible 1-$CP(i)$, where constraint (i) intersects each of the 2^κ $UHC[\bar{x}]$'s.

Proof: The proof will be by contradiction in a rather lengthy case analysis. In each case, we first assume the existence of an optimal solution $y \in \Re^2$ for (ILP) which is *not* a 1-$CP(i)$ with respect to any (i) intersecting a $UHC[\bar{x}]$. We then reach a contradiction by specifying a nonnegative integer solution x^* which is both feasible and strictly better than y in any possible configuration of $\bar{x}$, the binding constraints, y and c satisfying the original assumptions. Note that if $c = 0$ then all feasible integer solutions are optimal, not just 1-$CP(FR)$'s, so the theorem does not hold.

We first define a region which, under the assumptions about y, must contain only feasible integer solutions. The general picture to keep in mind is shown in Figure 3(a). The points $\{u_1, u_2, u_3, u_4\}$ are the all-integer vertices of $UHC[\bar{x}]$, while the points $\{v_1, v_2, v_3, v_4\}$ are the all-integer vertices of what Balas, *et al.* [BB] would refer to as the "dual to the unit hypercube centered at y," abbreviated here as $DUHC[y]$. Figure 3(a) shows a "feasibility cone" C_1 whose vertex is at $\bar{x}$ and whose edges are formed by constraints binding at $\bar{x}$. The edges of C_1 that will be used in the proof and that are shown in Figure 3(a) are the tightest possible "admissable binding constraints," *i.e.*, constraints binding at $\bar{x}$ which satisfy the

assumption that y is not a 1-$CP(i)$ with respect to any constraint $i \in \bar{A}$. Note that the edges of the cone C_1 shown in Figure 3(a) pass through the vertices of $DUHC[y]$ so that y is almost – but not quite – a 1-ceiling point with respect to each of the constraints binding at $\bar{x}$.

To further define the region of only feasible integer solutions, a second "feasibility cone" C_2 is constructed whose vertex is at y and whose edges are formed by constraints which do not intersect $UHC[\bar{x}]$. The edges of C_2 that will be used in the proof and that are shown in Figure 3(a) represent a limiting case for "admissable non-binding constraints," *i.e.*, constraints non-binding at $\bar{x}$ which neither chop off y nor intersect $UHC[\bar{x}]$. The tightest admissable non-binding constraints must actually form a cone strictly containing the cone C_2 pictured in Figure 3(a) because the edges of C_2 shown in the figure pass through the vertices of $UHC[\bar{x}]$, *i.e.*, they just barely intersect $UHC[\bar{x}]$. Thus, integer points that satisfy all of the constraints defining cones C_1 and C_2 are feasible for all configurations of the feasible region in which y is not a 1-ceiling point with respect to any constraint intersecting $UHC[\bar{x}]$.

Now a third cone C_3 is defined such that all points lying within this cone have at least as good an objective function value as that of y for all "admissable c," *i.e.*, for all c such that $\bar{x}$ is optimal for (LP_R). Cone C_3 comes from the Karush-Kuhn-Tucker necessary conditions for optimality in a linear program: at an optimal solution, the gradient of the objective function c can be expressed as a nonnegative combination of the gradients of the constraints binding at this solution. In other words, the cost vector c lies in the cone defined by the row vectors of A corresponding to the constraints binding at $\bar{x}$ [BJ, p. 215]. As a result, an "optimality cone" C_3 can be constructed such that any solution lying in this cone possesses an objective value greater than or equal to that of y for all admissable c. One edge of C_3 is formed by a constraint passing through y parallel to one edge of C_1 and the other edge is formed by another constraint passing through y parallel to the other edge of C_1, as illustrated in Figure 3(b).

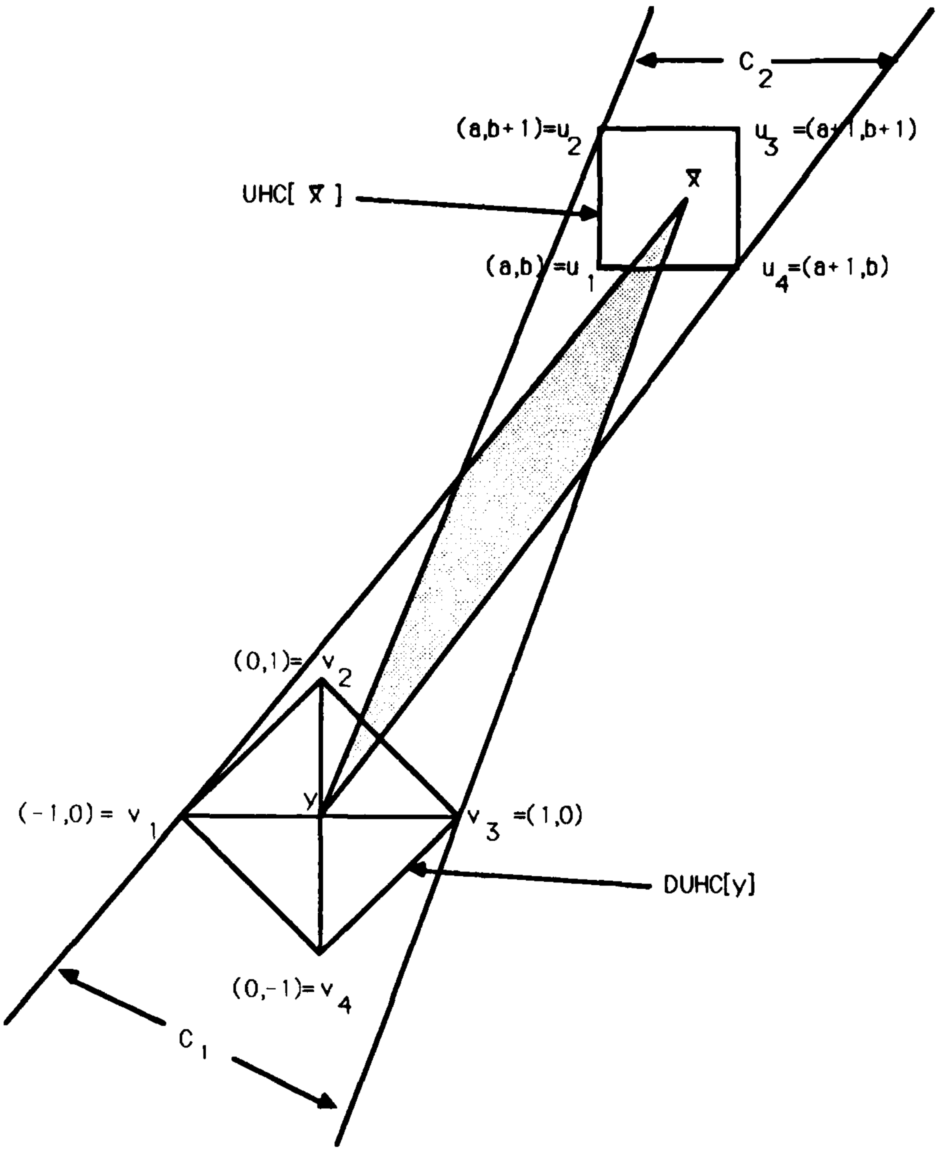

Figure 3(a). General setting for Theorem 4: Feasibility Cones C_1 and C_2.

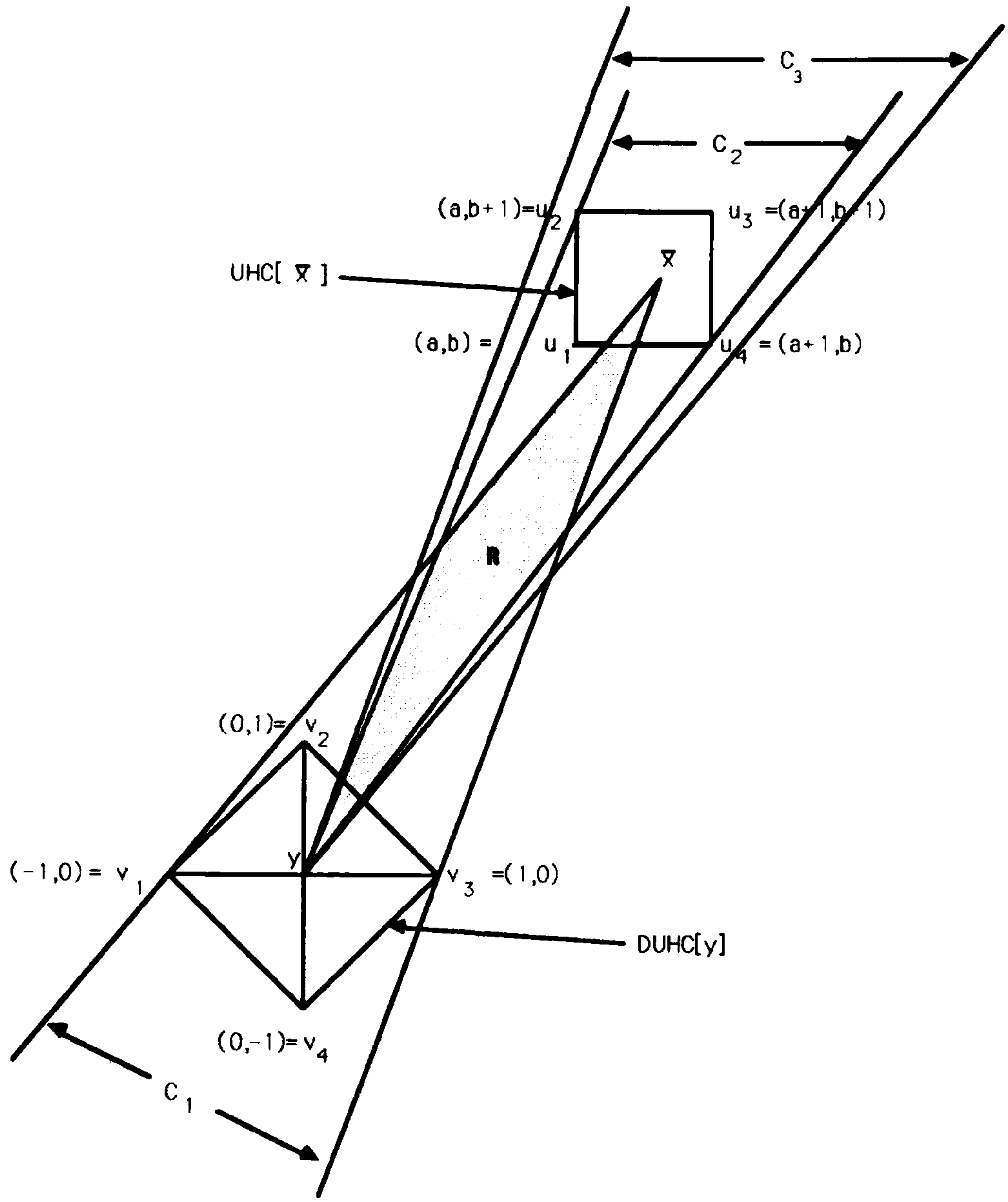

Figure 3(b). Theorem 4: Optimality Cone C_3.

Let region R denote the set of points that satisfy all six of the constraints defining cones C_1, C_2 and C_3, *i.e.*, $R \equiv (C_1 \cap C_2 \cap C_3)$. Since cones C_2 and C_3 share a common vertex, two of the constraints defining these cones must be redundant, so that only four constraints are really needed to define R. For example, in the arrangement shown in Figure 3(b), cone C_3 appears to be completely redundant with respect to C_2 (although this will not always be the case). Thus, all integer points in R are feasible and as good as y for all admissable c and cone constraints. We shall proceed to define a region $\bar{R} \subset R$ which possesses only solutions strictly better than y.

First, we account for the fact that $\bar{x}$ may lie anywhere within $UHC[\bar{x}]$. Consider the cone C_1' generated by locating $\bar{x}$ anywhere strictly interior to $UHC[\bar{x}]$ or on its northern or eastern boundary. This cone is at least as large as another cone C_1'' generated by locating $\bar{x}$ at some specific point along the southern or western boundary of $UHC[\bar{x}]$. Thus, it is sufficient to consider the case where $\bar{x}$ approaches some point along the southern or western boundary of $UHC[\bar{x}]$ because *the resultant cone is as small as possible*. In the following analysis, we will examine the southern and western boundaries of $UHC[\bar{x}]$ separately.

In Cases I.A and II.A, we consider locating $\bar{x}$ simultaneously at each point along the western edge of $UHC[\bar{x}]$, giving rise to a sequence of cones as pictured in Figure 3(c). For the moment, extend these cones only as far as vertices v_1 and v_3. Taking the intersection of all these truncated cones yields another truncated cone $\bar{C}_1$ with extreme points v, v_1 and v_3. For the case pictured in Figure 3(c), one edge of cone $\bar{C}_1$ passes through the points u_2 and v_3, while the other edge passes through the points u_1 and v_1. The sequence of C_1 cones, in turn, gives rise to a sequence of optimality cones. In the following figures, the cone labeled $\bar{C}_3$ is the intersection of all such optimality cones and the region $\bar{R} \equiv (\bar{C}_1 \cap C_2 \cap \bar{C}_3)$. Since C_2 shares a common vertex (y) with each optimality cone C_3, C_2 also shares a common vertex with $\bar{C}_3$. Thus, only four constraints are needed to define region $\bar{R}$.

Recall that $c^T x \geq c^T y, \forall x \in R$, with *equality* holding only when x lies on the boundary of cone C_3 and the optimal objective function hyperplane $\bar{z} = c^T x$ coincides with an edge

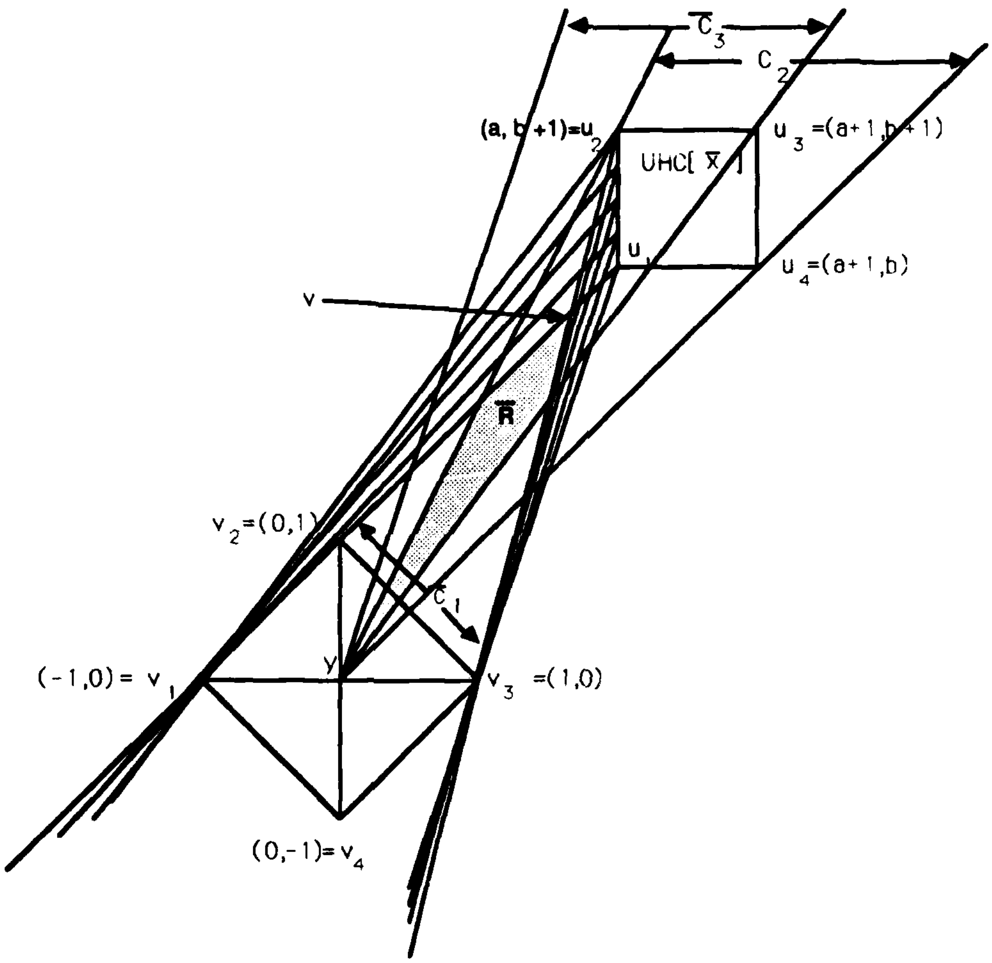

Figure 3(c). Cone $\bar{C}_1$ with vertex v is the intersection of a sequence of truncated C_1 cones whose vertices range between u_1 and u_2. Cone $\bar{C}_3$ with vertex y is the intersection of the corresponding sequence of C_3 optimality cones.

of C_3. An edge of C_3 may coincide with $\bar{z} = c^T x$ only if an edge of C_1 coincides with $\bar{z} = c^T x$ because of the way C_3's edges are constructed. However, the corresponding edges of cones $\bar{C}_1$ and $\bar{C}_3$ cannot be parallel because for every pair of parallel edges (one from C_1, one from C_3) at least one of the two edges becomes redundant when the intersection over all C_1 cones is taken to form $\bar{C}_1$ and the intersection over all C_3 cones is taken to form $\bar{C}_3$. Thus, no pair of parallel edges define $\bar{R}$. Consequently, $c^T x > c^T y, \forall x \in \bar{R}$, even if $x \in \bar{R}$ lies on the boundary of the optimality cone $\bar{C}_3$.

Cases I.B and II.B follow a plan similar to the above except that $\bar{x}$ is located simultaneously at each point along the *southern* edge of $UHC[\bar{x}]$. Taken together, these cases will show that a feasible integer solution x^* which is better than y can be identified in any admissible configuration of the $\bar{x}$, the binding constraints, y and c in $\Re^2$. To summarize, we will contradict the optimality of y by identifying an integer solution x^* in each case which lies in the region $\bar{R}$ defined by the following four constraints:

(1a) and (1b) - define the edges of the truncated feasibility cone $\bar{C}_1$,

(2) - defines one of the two edges of the feasibility cone C_2, and

(3) - defines one of the two edges of the optimality cone $\bar{C}_3$.

It is assumed that $y \leq \bar{x}$, but all the arguments could be appropriately rotated and a suitable x^* found, given any other relationship between y and $\bar{x}$ (*e.g.*, $y \geq \bar{x}$). Also, there is no loss of generality in letting y and u_1 be defined as the all-integer vertices $(0, 0)$ and (a, b), respectively. Finally, in what follows the notation $LHS(j)$ will mean the left hand side of the j^{th} constraint evaluated at x^*.

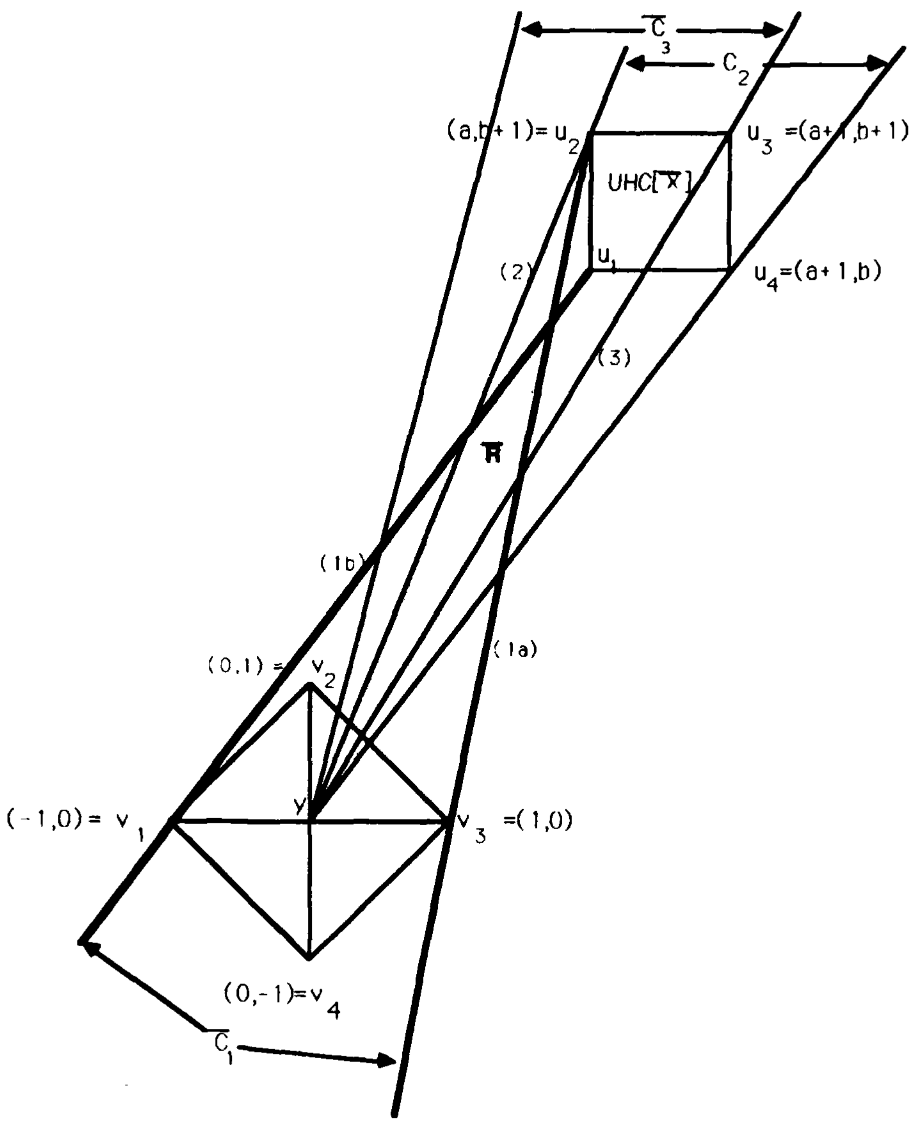

Figure 4(a). Case I.A of Theorem 4: $1 \leq a \leq b$; $\bar{x}$ on western edge of $UHC[\bar{x}]$.

Case I.A. $1 \le a \le b$; $\bar{x}$ **on western edge of** $UHC[\bar{x}]$. **See Figure 4(a).**

($1a$) Feasibility cone $\bar{C}_1$ constraint through (v_3, u_2) : $(b+1)x_1 - (a-1)x_2 \le b+1$

($1b$) Feasibility cone $\bar{C}_1$ constraint through (v_1, u_1) : $-bx_1 + (a+1)x_2 \le b$

(2) Feasibility cone C_2 constraint through (y, u_2) : $(b+1)x_1 - ax_2 \ge 0$

(3) Optimality cone $\bar{C}_3$ constraint through (y, u_3) : $(b+1)x_1 - (a+1)x_2 \le 0$

(i) a even: $\quad x^* \equiv (\lceil \frac{a}{2} \rceil, \lfloor \frac{b+1}{2} \rfloor) = (\frac{a}{2}, \lfloor \frac{b+1}{2} \rfloor)$

$LHS(1a) = (b+1)\frac{a}{2} - (a-1)\lfloor \frac{b+1}{2} \rfloor \le (b+1)\frac{a}{2} - (a-1)\frac{b}{2} = \frac{a+b}{2} \le b < b+1.$

$$LHS(1b) = -b\frac{a}{2} + (a+1)\lfloor \frac{b+1}{2} \rfloor = \begin{cases} \frac{-ab}{2} + \frac{(a+1)b}{2} = \frac{b}{2} \le b, & \text{(b even)} \\ \frac{-ab}{2} + \frac{(a+1)(b+1)}{2} = \frac{a+b+1}{2} \le b. & \text{(b odd)} \end{cases}$$

$LHS(2) = (b+1)\frac{a}{2} - a\lfloor \frac{b+1}{2} \rfloor \ge (b+1)\frac{a}{2} - a\frac{(b+1)}{2} = 0.$

Therefore, x^* is feasible.

$LHS(3) = (b+1)\frac{a}{2} - (a+1)\lfloor \frac{b+1}{2} \rfloor \le (b+1)\frac{a}{2} - (a+1)\frac{b}{2} = \frac{a-b}{2} \le 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

(ii) a odd: $\quad x^* \equiv (\lceil \frac{a}{2} \rceil, \lceil \frac{b+1}{2} \rceil) = (\frac{a+1}{2}, \lceil \frac{b+1}{2} \rceil)$

$LHS(1a) = (b+1)\frac{(a+1)}{2} - (a-1)\lceil \frac{b+1}{2} \rceil \le (b+1)\frac{(a+1)}{2} - (a-1)\frac{(b+2)}{2} = \frac{2b-a+3}{2} \le b+1.$

$$LHS(1b) = -b\frac{(a+1)}{2} + (a+1)\lceil \frac{b+1}{2} \rceil = \begin{cases} \frac{-ab-b}{2} + \frac{(a+1)(b+2)}{2} = a+1 \le b, & \text{(b even)} \\ \frac{-ab-b}{2} + \frac{(a+1)(b+1)}{2} = \frac{a+1}{2} \le b. & \text{(b odd)} \end{cases}$$

$LHS(2) = (b+1)\frac{(a+1)}{2} - a\lceil \frac{b+1}{2} \rceil \ge (b+1)\frac{(a+1)}{2} - a\frac{(b+2)}{2} = \frac{-a+b+1}{2} \ge 0.$

Therefore, x^* is feasible.

$LHS(3) = (b+1)\frac{(a+1)}{2} - (a+1)\lceil \frac{b+1}{2} \rceil = (a+1)\{\frac{b+1}{2} - \lceil \frac{b+1}{2} \rceil\} \le 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

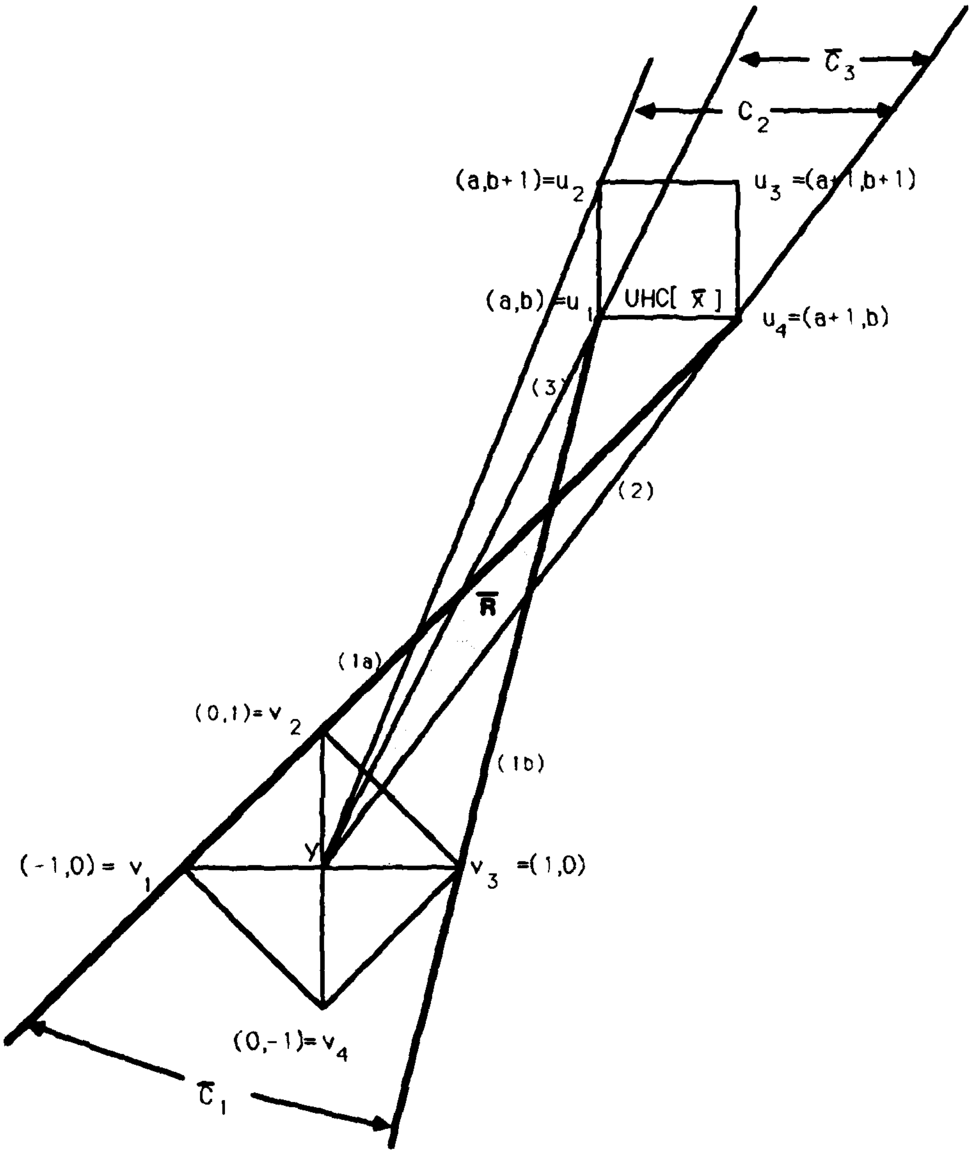

Figure 4(b). Case I.B of Theorem 4: $1 \leq a \leq b$; $\bar{x}$ on southern edge of $UHC[\bar{x}]$.

Case I.B. $1 \leq a \leq b$; $\bar{x}$ on southern edge of $UHC[\bar{x}]$. See Figure 4(b).

(1a) Feasibility cone $\bar{C}_1$ constraint through (v_1, u_4) : $-bx_1 + (a+2)x_2 \leq b$

(1b) Feasibility cone $\bar{C}_1$ constraint through (v_3, u_1) : $bx_1 - (a-1)x_2 \leq b$

(2) Feasibility cone C_2 constraint through (y, u_4) : $-bx_1 + (a+1)x_2 \geq 0$

(3) Optimality cone $\bar{C}_3$ constraint through (y, u_1) : $bx_1 - ax_2 \geq 0$

(i) a even, $a < b$: $x^* \equiv (\frac{a}{2}, \lfloor \frac{b}{2} \rfloor)$

$LHS(1a) = -b(\frac{a}{2}) + (a+2)\lfloor \frac{b}{2} \rfloor \leq -b(\frac{a}{2}) + (a+2)\frac{b}{2} = b.$

$LHS(1b) = b(\frac{a}{2}) - (a-1)\lfloor \frac{b}{2} \rfloor \leq \frac{ab}{2} - (a-1)\frac{(b-1)}{2} = \frac{a+b-1}{2} < b.$

$LHS(2) = -b(\frac{a}{2}) + (a+1)\lfloor \frac{b}{2} \rfloor \geq -\frac{ab}{2} + (a+1)\frac{b-1}{2} = \frac{-a+b-1}{2} \geq 0.$

Therefore, x^* is feasible.

$LHS(3) = b(\frac{a}{2}) - a\lfloor \frac{b}{2} \rfloor \geq a(\frac{b}{2} - \lfloor \frac{b}{2} \rfloor) \geq 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

(ii) a odd, $a < b$: $x^* \equiv (\frac{a+1}{2}, \lceil \frac{b}{2} \rceil)$

$$LHS(1a) = -b\frac{(a+1)}{2} + (a+2)\lceil \tfrac{b}{2} \rceil \leq \begin{cases} \frac{-b(a+1)}{2} + \frac{(a+2)(b+1)}{2} = \frac{a+b+2}{2} \leq b, & \text{(b odd)} \\ \frac{-b(a+1)}{2} + \frac{(a+2)b}{2} = \frac{b}{2} \leq b. & \text{(b even)} \end{cases}$$

$LHS(1b) = b\frac{(a+1)}{2} - (a-1)\lceil \frac{b}{2} \rceil \leq \frac{ab+b}{2} - (a-1)\frac{b}{2} = b.$

$LHS(2) = -b\frac{(a+1)}{2} + (a+1)\lceil \frac{b}{2} \rceil = (a+1)(\lceil \frac{b}{2} \rceil - \frac{b}{2}) \geq 0.$

Therefore, x^* is feasible.

$LHS(3) = b\frac{(a+1)}{2} - a\lceil \frac{b}{2} \rceil \geq (a+1)\frac{b}{2} - a\frac{(b+1)}{2}) = \frac{b-a}{2} \geq 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

(iii) $a = b$: $\quad x^* \equiv (\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor)$

$LHS(1a) = -b\lfloor \frac{a}{2} \rfloor + (a+2)\lfloor \frac{b}{2} \rfloor = -b\lfloor \frac{b}{2} \rfloor + (b+2)\lfloor \frac{b}{2} \rfloor = 2\lfloor \frac{b}{2} \rfloor \leq b.$

$LHS(1b) = b\lfloor \frac{a}{2} \rfloor - (a-1)\lfloor \frac{b}{2} \rfloor = \lfloor \frac{b}{2} \rfloor \leq b.$

$LHS(2) = -b\lfloor \frac{a}{2} \rfloor + (a+1)\lfloor \frac{b}{2} \rfloor = -b\lfloor \frac{b}{2} \rfloor + (b+1)\lfloor \frac{b}{2} \rfloor = \lfloor \frac{b}{2} \rfloor \geq 0.$

Therefore, x^* is feasible.

$LHS(3) = b\lfloor \frac{a}{2} \rfloor - a\lfloor \frac{b}{2} \rfloor = 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

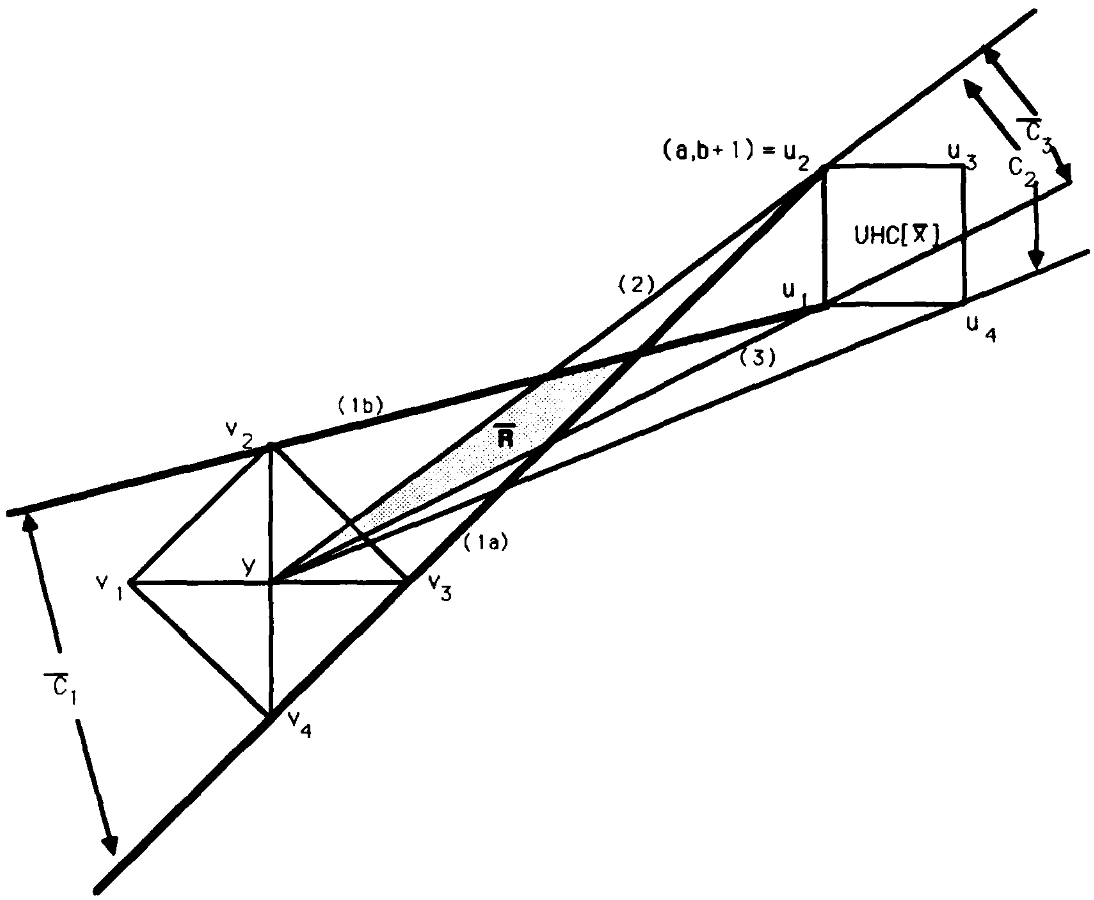

Figure 4(c). Case II.A of Theorem 4: $1 \leq b < a$; $\bar{x}$ on western edge of $UHC[\bar{x}]$.

Case II.A. $1 \leq b < a$; $\bar{x}$ on western edge of $UHC[\bar{x}]$. See Figure 4(c).

(1a) Feasibility cone $\bar{C}_1$ constraint through (v_4, u_2) : $(b+2)x_1 - ax_2 \leq a$

(1b) Feasibility cone $\bar{C}_1$ constraint through (v_2, u_1) : $-(b-1)x_1 + ax_2 \leq a$

(2) Feasibility cone C_2 constraint through (y, u_2) : $(b+1)x_1 - ax_2 \geq 0$

(3) Optimality cone $\bar{C}_3$ constraint through (y, u_1) : $bx_1 - ax_2 \leq 0$

(i) b even : $x^* \equiv (\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor) = (\lfloor \frac{a}{2} \rfloor, \frac{b}{2})$

$LHS(1a) = (b+2)\lfloor \frac{a}{2} \rfloor - \frac{ab}{2} \leq a\frac{(b+2)}{2}) - \frac{ab}{2} = a.$

$LHS(1b) = -(b-1)\lfloor \frac{a}{2} \rfloor + a(\frac{b}{2}) \leq -(b-1)\frac{a-1}{2} + \frac{ab}{2} = \frac{a+b-1}{2} < \frac{2a}{2} = a.$

$LHS(2) = (b+1)\lfloor \frac{a}{2} \rfloor - \frac{ab}{2} \geq (b+1)\frac{a}{2} + \frac{ab}{2} = \frac{-a}{2} \leq 0.$

Therefore, x^* is feasible.

$LHS(3) = b\lfloor \frac{a}{2} \rfloor - \frac{ab}{2} = b(\lfloor \frac{a}{2} \rfloor - \frac{a}{2}) \leq 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

(ii) b odd : $x^* \equiv (\lceil \frac{a}{2} \rceil, \lfloor \frac{b}{2} \rfloor) = (\lceil \frac{a}{2} \rceil, \frac{b}{2})$

$$LHS(1a) = (b+2)\lceil \tfrac{a}{2} \rceil - \tfrac{a(b+1)}{2} = (b+1)(\lceil \tfrac{a}{2} \rceil - \tfrac{a}{2}) + \lceil \tfrac{a}{2} \rceil = \begin{cases} \frac{a}{2} \leq a, & \text{(a even)} \\ \frac{a+b+2}{2} \leq a. & \text{(a odd)} \end{cases}$$

$LHS(1b) = -(b-1)\lceil \frac{a}{2} \rceil - a(\frac{b}{2}) \leq -\frac{a}{2}(b-1) + \frac{ab}{2} \leq a.$

$LHS(2) = (b+1)\lceil \frac{a}{2} \rceil - \frac{a(b+1)}{2} \geq (b+1)\frac{a}{2} - \frac{a(b+1)}{2} = 0.$

Therefore, x^* is feasible.

$LHS(3) = b\lceil \frac{a}{2} \rceil - \frac{a(b+1)}{2} \leq b\frac{(a+1)}{2} - \frac{a(b+1)}{2} = \frac{-a+b}{2} \leq 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

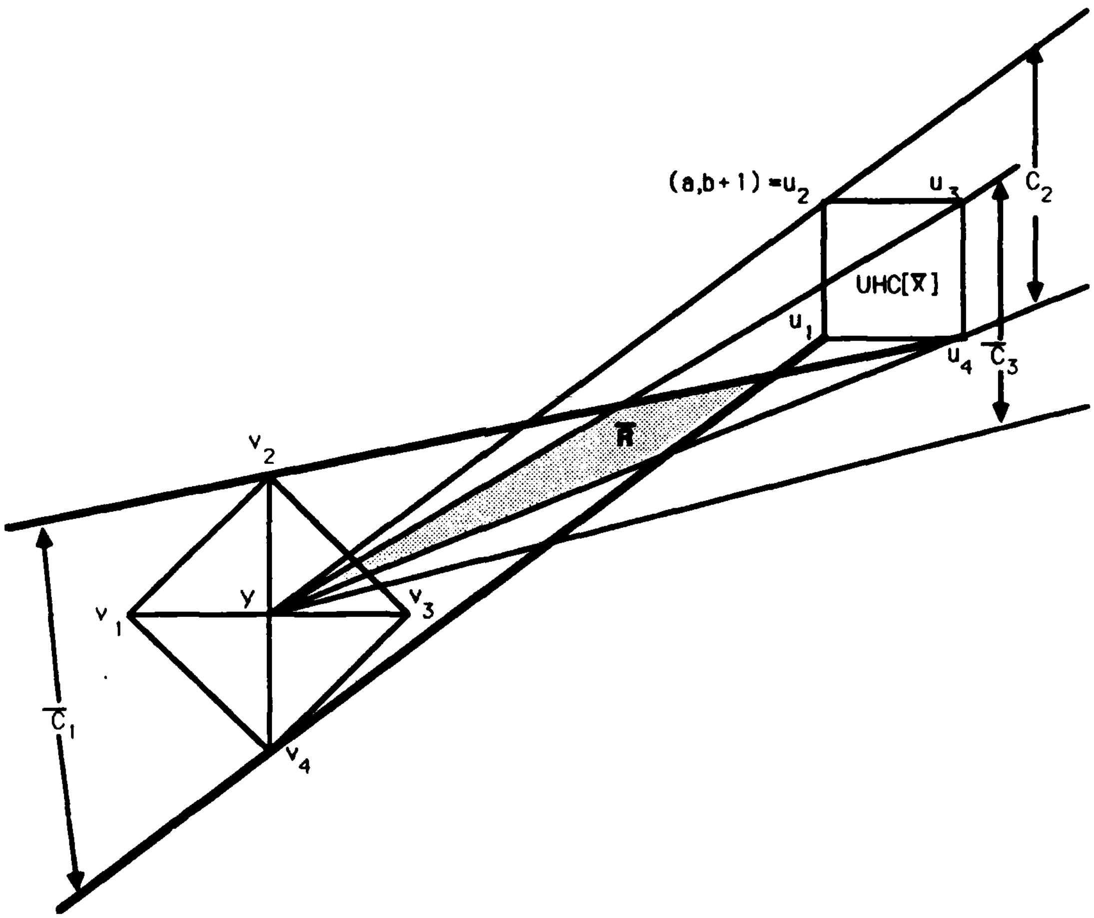

Figure 4(d). Case II.B of Theorem 4: $1 \leq b < a$; $\bar{x}$ on southern edge of $UHC[\bar{x}]$.

Case II.B. $1 \leq b < a$; $\bar{x}$ **on southern edge of** $UHC[\bar{x}]$. **See Figure 4(d).**

($1a$) Feasibility cone $\bar{C}_1$ constraint through (v_2, u_4): $-(b-1)x_1 + (a+1)x_2 \leq a+1$

($1b$) Feasibility cone $\bar{C}_1$ constraint through (v_4, u_1): $(b+1)x_1 - ax_2 \leq a$

(2) Feasibility cone C_2 constraint through (y, u_4): $-bx_1 + (a+1)x_2 \geq 0$

(3) Optimality cone $\bar{C}_3$ constraint through (y, u_3): $(b+1)x_1 - (a+1)x_2 \geq 0$

(i) b even : $\quad x^* \equiv (\lfloor \frac{a+1}{2} \rfloor, \lceil \frac{b}{2} \rceil) = (\lfloor \frac{a+1}{2} \rfloor, \frac{b}{2})$

$LHS(1a) = -(b-1)\lfloor \frac{a+1}{2} \rfloor + \frac{(a+1)b}{2} \leq -(b-1)\frac{a}{2} + \frac{(a+1)b}{2} = \frac{a+b}{2} < a+1.$

$LHS(1b) = (b+1)\lfloor \frac{a+1}{2} \rfloor - a(\frac{b}{2}) \leq (a+1)\frac{b+1}{2} - \frac{ab}{2} = \frac{a+b+1}{2} \leq a.$

$LHS(2) = -b\lfloor \frac{a+1}{2} \rfloor + \frac{(a+1)b}{2} \geq -b\frac{(a+1)}{2} + \frac{(a+1)b}{2} = 0.$

Therefore, x^* is feasible.

$LHS(3) = (b+1)\lfloor \frac{a+1}{2} \rfloor - \frac{(a+1)b}{2} \geq (b+1)\frac{a}{2} - \frac{(a+1)b}{2} = \frac{a-b}{2} > 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

(ii) b odd : $\quad x^* \equiv (\lceil \frac{a+1}{2} \rceil, \lceil \frac{b}{2} \rceil) = (\lceil \frac{a+1}{2} \rceil, \frac{b+1}{2})$

$LHS(1a) = -(b-1)\lceil \frac{a+1}{2} \rceil + \frac{(a+1)(b+1)}{2} \leq -(b-1)\frac{(a+1)}{2} + \frac{(a+1)(b+1)}{2} = a+1.$

$LHS(1b) = (b+1)\lceil \frac{a+1}{2} \rceil - \frac{a(b+1)}{2} \leq (b+1)\frac{(a+2)}{2} - (\frac{ab+a}{2}) = b+1 \leq a.$

$LHS(2) = -b\lceil \frac{a+1}{2} \rceil + \frac{(a+1)(b+1)}{2} \geq -b\frac{(a+2)}{2} + \frac{(a+1)(b+1)}{2} = \frac{a-b+1}{2} \geq 0.$

Therefore, x^* is feasible.

$LHS(3) = (b+1)\lceil \frac{a+1}{2} \rceil - \frac{(a+1)(b+1)}{2} \geq (b+1)\frac{(a+1)}{2} - \frac{(a+1)(b+1)}{2} = 0.$

Thus, x^* is strictly better than y, contradicting the assumption that y is optimal.

The final case that needs to be considered is where either $a = 0$ or $b = 0$ (but not both). If $a = 0$, the integer solution $x^* \equiv v_2 = (0,1)$ is clearly feasible and better than y, while if $b = 0$, the same can be said of $x^* \equiv v_3 = (1,0)$. ∎

Conjecture 4a (Theorem 4 with $n \geq 3$) Suppose an (ILP) is non-empty and bounded. Further suppose that $c \neq 0$ and $\bar{x}$ is an optimal solution for (LP_R). Then every optimal solution for (ILP) is a feasible 1-$CP(i)$, where constraint (i) intersects each of the 2^κ $UHC[\bar{x}]$'s.

Rationale: The reasoning behind this is that it seems the "geometry" of the situation in higher dimensions will reflect that of the two-dimensional case described in the proof of Theorem 4. For instance, projecting the situation in $\Re^3$ onto any of the two-dimensional coordinate planes x_1-x_2, x_2-x_3, or x_1-x_3 would reveal a feasible integer solution x^* in this projection with a value superior to that of y. Let p denote the midway point between $\bar{x}$ and y and $UHC[p]$ the unit hypercube containing p with all-integer vertices. For each case of Theorem 4, the solution x^* is the feasible integer solution closest to p, which is where the region defined by the three cones is widest. In higher dimensions, it seems likely that among the lattice points between y and $UHC[\bar{x}]$, one such integer solution which is approximately halfway between $\bar{x}$ and y will satisfy all the requirements for a feasible solution with an objective value greater than y for all admissable c, *i.e.*, all c such that $\bar{x}$ is optimal for (LP_R). First note that for some vertex v of $UHC[p]$, every component v_j of this vertex differs from p_j by no more than $\frac{1}{2}$. Then observe that, by construction, the width of the cone C_1 when sliced through p parallel to each coordinate axis is at least 1. Furthermore, the width of both cones C_2 and C_3 when sliced through p parallel to each coordinate axis is at least $\frac{1}{2}$. So, in the neighborhood of p, it appears possible to move to an integer solution x^* (a vertex of $UHC[p]$) which satisfies the constraints defining all three cones. Thus, it seems plausible in higher dimensions that we will be able to find a feasible integer solution x^* better than y when we assume that y is not a 1-$CP(i)$ with

respect to any constraint which intersects one the 2^{κ} $UHC[\bar{x}]$'s. ▮

Consider the implications for developing an algorithm to find an optimal solution for (ILP) if this conjecture is correct. The conjecture implies that the set of functional constraints whose 1-$CP(i)$'s we must enumerate in order to solve (ILP) consists of just those which intersect a unit hypercube with all-integer vertices containing $\bar{x}$. Part of this set of constraints, namely, the set $\bar{A}$ of constraints binding at $\bar{x}$, may be identified by using the simplex method to solve the (LP_R). The next step would be to pivot from $\bar{x}$ to find all extreme points adjacent to it and check whether or not each adjacent extreme point is contained in $UHC[\bar{x}]$; if it is, then any constraint binding at this extreme point intersects $UHC[\bar{x}]$. We still would need to test whether or not any of the remaining constraints intersect $UHC[\bar{x}]$. A sufficient condition for excluding a constraint from the set is if the distance from the constraint to $\bar{x}$ at its closest point is more than $\sqrt{n}$. Such a constraint cannot possibly intersect $UHC[\bar{x}]$. Constraints less than a distance of $\sqrt{n}$ from $\bar{x}$ at their closest point do not neccessarily intersect $UHC[\bar{x}]$, but to determine this precisely takes more effort. However, computational experience has shown that it is unnecessary to determine the complete set of constraints intersecting $UHC[\bar{x}]$ because the problem is usually solved by the time we have enumerated the 1-ceiling points with respect to just a few constraints binding at $\bar{x}$. Thus, while Theorem 4 and Conjecture 4a are of theoretical interest, they have had little impact on the development of the Exact Ceiling Point Algorithm described in a subsequent report.

5. Underlying Assumptions

The theorems of the previous sections imply that to solve a non-trivial, non-empty, bounded (ILP) with $c \neq 0$, it is sufficient to enumerate its feasible 1-$CP(i)$'s. In this section we consider the effect of placing these conditions on (ILP). First, assuming that $c \neq 0$ just means that we have an optimization problem. Second, to check that (ILP) is not trivial to solve, we simply examine some of the data; if both $c \leq 0$ and $b \geq 0$, then $x = 0$ trivially solves the (ILP). Third, although we can conclude that (ILP) is empty if the associated (LP_R) is empty, the contrapositive is not true. So, when (LP_R) is non-empty, we can determine whether or not (ILP) is empty by seeking only 1-$CP(FR)$'s because an (ILP) containing no 1-ceiling points does not contain any feasible integer solutions either, and vice-versa, as Theorem 5 demonstrates. Theorem 5 follows directly from the next two lemmas.

Lemma 5a The set of feasible integer solutions for (ILP) is empty if and only if the set of $CP(FR)$'s is empty.

Proof: $(\Rightarrow)$ Assume no $CP(FR)$'s exist. Then, by Lemma 1a, no extreme points of the convex hull of feasible integer solutions exist. Since the feasible region for (ILP) is bounded below, the convex hull of feasible integer solutions must be empty (rather than being unbounded). Therefore, no feasible integer solutions exist.

$(\Leftarrow)$ Assume no feasible integer solutions exist. Then, since every $CP(FR)$ is a special kind of feasible integer solution, no $CP(FR)$'s may exist either. ∎

Lemma 5b The set of $CP(FR)$'s for (ILP) is empty if and only if the set of 1-$CP(FR)$'s is empty.

Proof: $(\Rightarrow)$ Assume no 1-$CP(FR)$'s exist. Since every $CP(FR)$ is a special kind of feasible 1-$CP(FR)$ by Lemma 1c, part (a), no $CP(FR)$'s may exist either.

($\Leftarrow$) Assume no $CP(FR)$'s exist. Then, by Lemma 5a, the set of feasible integer solutions for (ILP) is empty and thus no 1-$CP(FR)$'s exist. ∎

Theorem 5 The set of feasible integer solutions for (ILP) is empty if and only if the set of 1-$CP(FR)$'s is empty.

Proof: Immediate from Lemmas 5a and 5b. ∎

Finally, the boundedness condition can also be checked by solving the linear programming relaxation of (ILP) and possibly enumerating 1-ceiling points.

Theorem 6 If the objective function of (LP_R) is bounded above, then either the objective function of (ILP) is also bounded above or (ILP) is infeasible.

Proof: Even if solutions exist for (LP_R), infeasiblity of (ILP) is clearly a possibility. For example, the region defined by $\{x \mid 1 \leq 3x_j \leq 2, \forall j\}$ contains real but no integer solutions. Because (LP_R) is a relaxation of (ILP) the optimal objective function value of (ILP) must be less than or equal to that of (LP_R). Thus, if the former is bounded above, the latter must also be bounded above. ∎

When the objective function of (LP_R) is unbounded and all data are rational, we can use the following result (see [Pa, Lemma 3] or [BG, Theorem 1]).

Theorem 7 (Papadimitriou) Assume the data of (ILP) are rational. If the objective function of (LP_R) is unbounded above, then either the objective function of (ILP) is also unbounded above or (ILP) is infeasible.

Whether (LP_R) is bounded or not, we may use an algorithm which solves the linear programming relaxation and searches only for 1-ceiling points of (ILP) to make a relevant statement about the optimal value of (ILP). To summarize: (1) if (LP_R) is infeasible,

so is (ILP); (2) if the objective function of (LP_R) is bounded above, then a 1-ceiling point algorithm applied to (ILP) will determine either that (ILP) has no feasible integer solutions or that it also has an objective function which is bounded above; (3) if the objective function of (LP_R) is unbounded, then a 1-ceiling point algorithm applied to (ILP) will conclude either that (ILP) has no feasible solutions or, upon first identifying a feasible integer solution, that (ILP)'s objective function is also unbounded above. In this latter case, the integer linear program has most likely been formulated incorrectly.

6. Summary of Concepts with an Example

We shall now briefly review the ideas presented in this report and try to illustrate their effects graphically. The key theorems given above will be applied in turn to the simple example of Figure 1 having two variables and three constraints. After each is applied, the set of integer solutions on which we need to focus in order to solve the problem, shown as the solid lattice points in Figures 5 and 6, is reduced. The first diagram below, Figure 5(a), illustrates that initially all feasible integer solutions of our (ILP) are under consideration. The significance of Theorem 2 is that to solve our problem, we need only to focus on the feasible 1-$CP(i)$'s, as shown in Figure 5(b).

Considering Theorem 3, we need retain from the set of all feasible 1-$CP(i)$'s only those which are 1-ceiling points with respect to a *functional* constraint. Therefore, we can ignore those integer points which lie along either of the axes but which are either infeasible or are not 1-ceiling points with respect to a functional constraint. The effect of this is shown in Figure 6(a), where the set of 1-ceiling points $\{(x_1, 0), x_1 = 0, 1, ..., 4\}$ are dropped from consideration.

Finally, if we solve the linear programming relaxation associated with (ILP), we may take one more step based on its optimal solution, $\bar{x}$. From the remaining set of points,

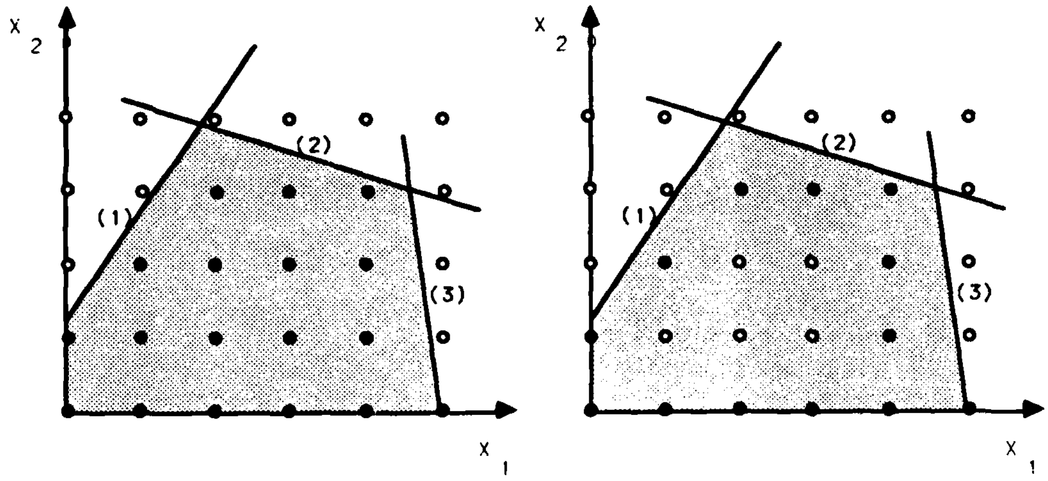

Figure 5.(a) All feasible solutions. (b) Feasible 1-$CP(i)$'s (Theorem 2).

Theorem 4 allows us to just concentrate on those which are 1-ceiling points with respect to a functional constraint that intersects a unit cube containing $\bar{x}$ and possessing all-integer vertices. If we assume that $\bar{x}$ occurs at the intersection of constraints (1) and (2), then we no longer need to examine most of the feasible 1-$CP(3)$'s, namely, $\{(4, x_2), x_2 = 0, 1, 2\}$, because they are not 1-$CP(i)$'s with respect to either constraint (1) or constraint (2). The remaining integer points which are still sufficient to solve (ILP) are shown in Figure 6(b).

Thus, the net effect of the theorems is to significantly reduce the set of feasible integer solutions on which we need to focus our attention in order to solve (ILP), assuming that feasible solutions exist. Subsequent reports will describe a heuristic algorithm and an exact algorithm, respectively, which are designed to take advantage of these ideas.

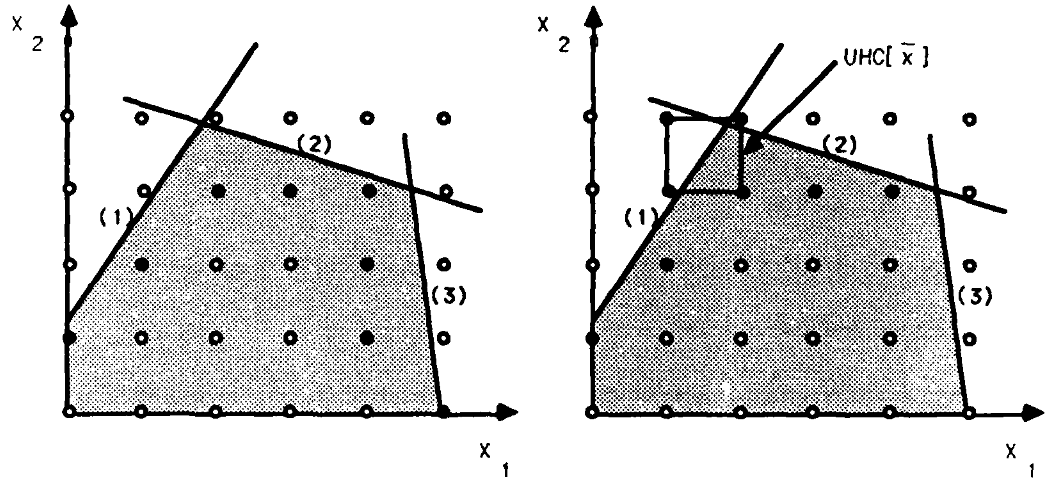

Figure 6.(a) Effect of Theorem 3. (b) Effect of Theorem 4.

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SOL 88-11: The Role of Ceiling Points in General Integer Linear Programming, Robert M. Saltzman and Frederick S. Hillier (July 1988, 35 pp.)

This report examines the role played by several kinds of ceiling points in solving the pure, general integer linear programming problem *(ILP)*. While no assumptions are made concerning the structure or signs of the data of the problem, it is assumed that the feasible region for *(ILP)* is non-empty and bounded. A ceiling point with respect to a single constraint may be thought of as an integer solution on or close to the boundary of the feasible region defined by the constraint. The definition of a ceiling point with respect to a single constraint is extended to take multiple constraints into consideration simultaneously, defining what is called a feasible ceiling point. It is shown that the set of all feasible ceiling points contains at least one optimal solution for *(ILP)*. A related class of solutions called feasible 1-ceiling points is also characterized and shown to contain *all* optimal solutions for *(ILP)*. Moreover, 1-ceiling points are computationally easier to identify than ordinary ceiling points and may be sought with respect to one constaint at a time. It is also demonstrated that solving *(ILP)* requires only enumerating feasible 1-ceiling points with respect to a subset of all functional constraints.